JOURNAL OF APPROXIMATION THEORY **10**, 245–267 (1974)

On Some Geometric Properties of Suns

BRUNO BROSOWSKI

Gesellschaft für wissenschaftliche Datenverarbeitung m.b.H. Göttingen, 34 Göttingen-Nikolausberg, Am Fassberg, West Germany

AND

FRANK DEUTSCH

Department of Mathematics, The Pennsylvania State University, University Park, Pennsylvania 16802

Communicated by L. Collatz

1. INTRODUCTION

The concept of a sun, introduced by Efimov and Steckin in [10], has proved to be rather important in the general theory of approximation in normed linear spaces. (Recall that a subset V of a normed linear space X is called a sun if, whenever $v_0 \in V$ is a best approximation to some $x \in X$, then v_0 is also a best approximation to every point on the ray from v_0 through x.) We mention only the following results:

(1) Every convex set is a sun and in smooth spaces every proximinal sun is convex.

(2) In finite-dimensional normed linear spaces every Chebyshev set is a sun (cf. Vlasov [14]).

But it is still an open problem whether a Chebyshev set in an arbitrary normed linear space must be a sun.[1] In [6, 7] the concept of a Kolmogorov set was introduced and it was observed that these sets are equivalent to suns. A subset V of a normed linear space is called a Kolmogorov set if we have $V \cap K(v_0, f) = \varnothing$ for every v_0 in $P_V(f)$. Here $P_V(f)$ denotes the set of best approximations of f by means of the elements of V and $K(v_0, f)$ denotes the cone

$$K(v_0, f) := \{v \in X\colon \operatorname{Re} x^*(v - v_0) > 0 \text{ for each } x^* \in \mathscr{E}(f - v_0)\}$$

[1] Dunham has recently given an example of a Chebyshev subset in $C[0, 1]$ which is not a sun.

and $\mathscr{E}(f - v_0)$ denotes the set of extremal points of the set

$$\{x^* \in X^*: \| x^* \| \leq 1 \text{ and } x^*(f - v_0) = \| f - v_0 \|\}.$$

It is now known (cf. [4, 5] and the bibliography cited there) that much of the linear or convex approximation theory can be extended to approximation by elements of suns. In [5] there was given a characterization of suns by intrinsic properties (i.e., without referring to an approximation problem) which is rather complicated mainly due to the fact that for characterizing the suns geometrically one uses not only the linear functionals in $\mathscr{E}(f - v_0)$ but also the functionals of the σ_{E_p}-closed sets containing $\mathscr{E}(f - v_0)$ (cf. [5] for details). This is in contrast to earlier results [3, 4] in the space $C(T, H)$, i.e., the space of all continuous mappings from a compact Hausdorff-space T into a pre-Hilbert-space H and endowed with the Chebyshev-norm. In this special case we have the following characterization (cf. [3, 4]): A subset V of $C(T, H)$ is a sun if and only if for each f in $C(T, H)$ we have $K(f, v_0) \cap V \neq \varnothing$ implies v_0 is in the closure of the set

$$\{v \in V: (f(t) - v_0(t), v(t) - v_0(t)) > \tfrac{1}{2} \| v(t) - v_0(t)\|^2 \text{ for every } t \text{ in } \operatorname{crit}(f - v_0)\}.$$

Here $\operatorname{crit}(f - v_0)$ denotes the set

$$\{t \in T: \| f - v_0 \| = \| f(t) - v_0(t)\|_H\}.$$

In the special case $H = \mathbb{R}$ this condition simplifies to: A subset V of $C(T, \mathbb{R})$ is a sun if and only if for each f in $C(T, \mathbb{R})$ we have $K(f, v_0) \cap V \neq \varnothing$ implies v_0 in the closure of the set

$$\{v \in V: (f(t) - v_0(t)) \cdot (v(t) - v_0(t)) > 0 \text{ for every } t \text{ in } \operatorname{crit}(f - v_0)\}.$$

Taking in account the representation of the extremal functionals on $C(T, H)$ we see that for characterizing the suns in $C(T, H)$ we need only the linear functionals in $\mathscr{E}(f - v_0)$. The condition valid in the special case $C(T, \mathbb{R})$ is easy generalized to arbitrary normed linear spaces, namely,

$$\text{(M)} \qquad K(f, v_0) \cap V \neq \varnothing \Rightarrow v_0 \in \overline{K(f, v_0) \cap V}.$$

It is easy to see that a sun in any normed linear space satisfies this condition (M). There arises the question whether there are spaces other than $C(T, \mathbb{R})$ where this condition (M) is always sufficient for a set V to be sun. We will call such spaces MS-spaces. This problem was investigated first in [1]. It was shown there that a geometric property of the unit sphere ("strong nonlunarity") was sufficient in order that every set with property (M) (such sets are called moons in [1]) be a sun. In particular, the following spaces are strongly nonlunar [1]:

(a) the space $C_0(T)$, T locally compact Hausdorff;

(b) the space $L_1(S, \Sigma, \mu)$, (S, Σ, μ) a σ-finite measure space, if and only if (S, Σ, μ) is purely atomic;

(c) every finite-dimensional space whose unit-ball is a polyhedron.

It is not known whether every MS-space is strongly nonlunar.

In this work we continue the study begun in [1]. In §2 we first derive some properties of suns. The main result in this part is the following:

THEOREM. *Let V be a subset of a real normed linear space X. Then we have the implications* "$(i) \Rightarrow (i + 1)$", $i = 1, 2, 3$,

(1) *V is a sun;*

(2) *the metric projection P_V associated with V is* ORL-*continuous* (*see definition below*);

(3) *for each f in X, every local minimum of the functional*

$$\Phi_f(v) := \| f - v \|$$

on V is a global minimum;

(4) *V is a moon.*

In a MS-*space all these conditions are obviously equivalent.*

Here ORL-continuity refers to a generalization of the usual notion of lower semi-continuity for set-valued mappings (see §2 for a precise definition). Roughly speaking the metric projection is ORL-continuous at a given point if it is lower semi-continuous as we approach the point from a certain direction along prescribed lines. It should be remarked that the metric projection associated with a sun in general is not continuous even in MS-spaces, as the results of Werner [15] show in the case of rational functions in the space $C[a, b]$. Perhaps surprising, then, is in MS-spaces suns are characterized by the ORL-continuity of their metric projections. (It is not known whether this result is valid in non-MS-spaces, e.g., in strictly convex spaces.) An immediate consequence is the result that in a MS-space every Chebyshev set with ORL-continuous metric projection is a sun. In §§3 and 4 we describe some constructions to get new MS-spaces from given ones. For this purpose we define in §2 the notion of an (A)-space, which is (formally at least) a strengthening of strong nonlunarity. We get the following theorems:

(1) *If T is locally compact Hausdorff and X an* (A)-*space* (*with a certain additional property*), *then the space $C_0(T, X)$ is an* (A)-*space.*

(2) *If X_i, i in I, is a family of* (A)-*spaces then the c_0-product is an* (A)-*space; and the l_1-product is an* (A)-*space provided each X_i satisfies a certain additional property.*

During the course of the proof of the first of these two theorems we obtain—in collaboration with P. D. Morris—a representation of the extreme points of the unit ball in the dual of $C_0(T, X)$. This generalizes a result of Singer [12], who proved the special case when T is compact and X a Banach space using more elaborate machinery.

2. General Results

Throughout this paper, unless specified otherwise, X will denote a real normed linear space, X^* its dual space, $B(X) = \{x \in X: \| x \| \leq 1\}$, $S(X) = \{x \in X: \| x \| = 1\}$, $B(x, \epsilon) = \{y \in X: \| x - y \| < \epsilon\}$, and $\mathscr{E}(X)$ is the set of extreme points of $B(X)$. For each $x \in X$, we denote the extreme points of the peaking set for x by $\mathscr{E}(x)$. Thus

$$\mathscr{E}(x) = \{x^* \in \mathscr{E}(X^*): x^*(x) = \| x \|\}.$$

By the *σ-topology* on $\mathscr{E}(X^*)$, we will mean the relative weak* topology on $\mathscr{E}(X^*)$. For a given set V in X, the *metric projection* onto V is the set-valued mapping P_V defined on X by

$$P_V(x) = \{v \in V: \| x - v \| = \text{dist}(x, V)\}.$$

For an element $v_0 \in V$, the inverse of P_V is given by

$$P_V^{-1}(v_0) = \{x \in X: v_0 \in P_V(x)\}.$$

If $P_V(x) \neq \varnothing$ (respectively, $P_V(x)$ is a singleton) for each $x \in X$, V is called a *proximinal* (respectively, *Chebyshev*) set.

All other undefined notation or terminology can be found in [9].

Definition 2.1. X is called an (A)-space if for each nonzero x in X and each σ-open subset $W \supset \mathscr{E}(x)$, there exists y in X such that

$$W \supset \mathscr{E}(y) \tag{1}$$

and

$$\sup\{x^*(x): x^* \in \mathscr{E}(X^*) \backslash \mathscr{E}(y)\} < \| x \|. \tag{2}$$

Remark. We note that condition (2) implies that $\mathscr{E}(y) \supset \mathscr{E}(x)$. Also, by Lemma 2 in [7], conditions (1) and (2) are equivalent to the existence of $z \in X$ such that

$$W \supset \mathring{\mathscr{E}}(z) \tag{1'}$$

and

$$\sup\{x^*(x): x^* \in \mathscr{E}(X^*) \backslash \mathring{\mathscr{E}}(z)\} < \| x \|, \tag{2'}$$

where $\mathring{\mathscr{E}}(z)$ denotes the σ-interior of $\mathscr{E}(z)$. (This formulation of (A)-space will be useful in the proof of Theorem 3.1.)

As an immediate consequence of the definition, we have (taking y equal to x):

LEMMA 2.2. *If, for each* $x \in S(X)$,

$$\sup\{x^*(x): x^* \in \mathscr{E}(X^*)\backslash\mathscr{E}(x)\} < 1,$$

then X *is an* (A)-*space.*

COROLLARY 2.3. *All finite-dimensional polyhedral spaces* (*i.e., those whose unit balls are polygons*) *and* c_0 (*or more generally* $c_0(T)$ *for any index set* T) *are* (A)-*spaces.*

Proof. In both cases, for each $x \in S(X)$, the set of all $x^* \in \mathscr{E}(X^*)\backslash\mathscr{E}(x)$ such that $\frac{1}{2} \| x \| < x^*(x)$ is finite.

Remark. The condition of Lemma 2.2 is not necessary in general for (A)-spaces. This can be easily verified by considering the (A)-space l_1 and noting that

$$\sup\{x^*(x): x^* \in \mathscr{E}(X^*)\backslash\mathscr{E}(x)\} = 1$$

for every $x \in S(l_1)$ which has infinitely many nonzero components.

We recall the following definition. For any pair of points v_0, x in X, we define an open cone

$$K(v_0, x) = \{y \in X: x^*(y - v_0) < 0 \text{ for every } x^* \in \mathscr{E}(v_0 - x)\}.$$

The space X is called *strongly nonlunar* [1] if, for each $v_0 \in S(X)$ and $u \in \overline{K(v_0, 0)}$, there exists an $x \in X$ such that $u \in K(v_0, x)$ and $v_0 \notin \overline{K(v_0, x) \cap S(X)}$. (Actually, in [1] the point x was supposed to lie in $B(0, 1)$. However, it is easy to show that this definition is equivalent to the original one.)

THEOREM 2.4. *Every* (A)-*space is strongly nonlunar.*

Proof. Let X be an (A)-space, $v_0 \in S(X)$, and $u \in K(v_0, 0)$, i.e., $\sup\{x^*(u): x^* \in \mathscr{E}(v_0)\} = 1 - \delta$ for some $\delta > 0$. Let

$$W = \left\{x^* \in \mathscr{E}(X^*): 1 - \frac{\delta}{2} < x^*(v_0),\ x^*(u) < 1 - \frac{\delta}{2}\right\}.$$

Then W is σ-open and $W \supset \mathscr{E}(v_0)$. Choose $y \in X$ such that $\mathscr{E}(y) \subset W$ and

$$\sup\{x^*(v_0): x^* \in \mathscr{E}(X^*)\backslash\mathscr{E}(y)\} = 1 - \epsilon$$

for some $\epsilon > 0$. Setting $x = v_0 - y$, we see that $W \supset \mathscr{E}(v_0 - x)$ and

$$\sup\{x^*(v_0): x^* \in \mathscr{E}(X^*)\backslash\mathscr{E}(v_0 - x)\} = 1 - \epsilon.$$

If $x^* \in \mathscr{E}(v_0 - x)$, then $x^* \in W$ and

$$x^*(u - v_0) = x^*(u) - x^*(v_0) < 1 - \frac{\delta}{2} - \left(1 - \frac{\delta}{2}\right) = 0.$$

Thus $u \in K(v_0, x)$. Now suppose $v \in B(v_0, \epsilon) \cap K(v_0, x)$. If $x^* \in \mathscr{E}(v_0 - x)$, $x^*(v) < x^*(v_0) \leqslant 1$. If $x^* \in \mathscr{E}(X^*)\backslash\mathscr{E}(v_0 - x)$, then

$$x^*(v) = x^*(v - v_0) + x^*(v_0) \leqslant \| v - v_0 \| + 1 - \epsilon < 1.$$

Thus $\| v \| < 1$. We have shown that $B(v_0, \epsilon) \cap K(v_0, x) \subset B(0, 1)$, i.e., $v_0 \notin \overline{K(v_0, x) \cap S(X)}$. This completes the proof.

Let $V \subset X$. A point $v_0 \in V$ is called a *lunar point* of V if $x \in P_V^{-1}(v_0)$ and $K(v_0, x) \cap V \neq \varnothing$ imply $v_0 \in \overline{K(v_0, x) \cap V}$. V is called a *moon* [1] if each of its points are lunar. V is called a *sun* [10] if, for each $v_0 \in V$, $x \in P_V^{-1}(v_0)$ implies $v_0 + \lambda(x - v_0) \in P_V^{-1}(v_0)$ for every $\lambda \geqslant 0$.

V is called a *Kolmogorov set* [5] if, for each $v_0 \in V$, $x \in P_V^{-1}(v_0)$ implies

$$\min\{x^*(v - v_0): x^* \in \mathscr{E}(x - v_0)\} \leqslant 0$$

for every $v \in V$.

It is obvious that every sun is a moon. Further, we have:

THEOREM 2.5. *Let $V \subset X$. The following are equivalent*:

(1) *V is a sun.*

(2) *V is a Kolmogorov set.*

(3) *For each $v_0 \in V$, $K(v_0, x) \cap V = \varnothing$ for every $x \in P_V^{-1}(v_0)$.*

The equivalence of (1) and (2) was first given in [5], while the equivalence of (1) and (3) (as well as an alternate proof of the equivalence of (1) and (2)) was given in [1]. From the results of [1], we cite the following:

(a) if X is strongly nonlunar, then a set in X is a sun if and only if it is a moon;

(b) the space $C_0(T)$, T locally compact Hausdorff, is strongly nonlunar;

(c) the space $L_1(S, \Sigma, \mu)$, (S, Σ, μ) a σ-finite measure space, is strongly nonlunar if and only if (S, Σ, μ) is purely atomic.

The converse of (a) is still an open question, i.e., if every moon in X is a sun, must X be strongly nonlunar? Relative to this problem we can prove the following:

THEOREM 2.6. *Let X be a normed linear space such that every moon is a sun. Then each point of S(X) is not a lunar point.*

Proof. If some $v_0 \in S(X)$ were a lunar point, then $v_0 \in \overline{K(v_0, x) \cap S(X)}$ for every $x \in P_{S(X)}^{-1}(v_0) \cap B(0, 1)$. Let $V = [X \backslash B(X)] \cup \{v_0\}$. V is obviously not a sun. We will show V is a moon. For this it suffices to check only the point $v_0 \in V$ (since every other $v \in V$ is interior to V so $v = P_V^{-1}(v)$ and $K(v, v) = \varnothing$). Let $x \in P_V^{-1}(v_0)$, $x \neq v_0$. Then $x \in B(0, 1)$ and so $V_0 \in \overline{K(v_0, x) \cap S(X)}$. Thus for each $\epsilon > 0$ there exists $v_\epsilon \in K(v_0, x) \cap S(X)$ such that $\| v_0 - v_\epsilon \| < \epsilon/2$. Since $K(v_0, x)$ is open, there exists $u_\epsilon \in K(v_0, x)$ and $\| u_\epsilon \| > 1$ such that $\| u_\epsilon - v_\epsilon \| < \epsilon/2$. Then $u_\epsilon \in V$ and $\| u_\epsilon - v_0 \| < \epsilon$ so $v_0 \in \overline{K(v_0, x) \cap V}$. Thus V is a moon.

Let $\mathscr{P}[0, 1]$ denote the space of all polynomials on $[0, 1]$ endowed with the supremum norm.

COROLLARY 2.7. *In $\mathscr{P}[0, 1]$, there is a moon which is not a sun.*

Proof. Let $X = \mathscr{P}[0, 1]$. It suffices to show that the function $v_0(t) = -1 + 2t$ is a lunar point of $S(X)$. Let $x \in B(0, 1)$. It follows that $v_0 - x$ is a nonconstant polynomial. Using the well-known fact that $\mathscr{E}(X^*)$ consists of (plus and minus) all point evaluations, we see that

$$\begin{aligned} K(v_0, x) = \{v \in X \colon & v(t) < v_0(t) \text{ if } v_0(t) - x(t) = \| v_0 - x \|, \\ & v(t) > v_0(t) \text{ if } v_0(t) - x(t) = -\| v_0 - x \|\}. \end{aligned}$$

Given any $\epsilon > 0$, we can "perturb" v_0 by an amount less that $\epsilon/2$ at each point to obtain a continuous function $\tilde{v}_0$ such that $\tilde{v}_0(t) < v_0(t)$ for all (finitely many) t such that $v_0(t) - x(t) = \| v_0 - x \|$, $\tilde{v}_0(t) > v_0(t)$ for every t such that $v_0(t) - x(t) = -\| v_0 - x \|$, and $\| \tilde{v}_0 \| = 1$. By a result of Wolibner [16], we can choose a polynomial v such that $\| v - \tilde{v}_0 \| < \epsilon/2$, $v(t) = \tilde{v}_0(t)$ for all t such that $| v_0(t) - x(t)| = \| v_0 - x \|$, and $\| v \| = 1$. Thus $\| v - v_0 \| < \epsilon$ and $v \in K(v_0, x) \cap S(X)$. Hence $v_0 \in \overline{K(v_0, x) \cap S(X)}$ and v_0 is a lunar point of $S(X)$.

Remark. The same argument as in the corollary shows that, in the space of analytic functions on $[0, 1]$, there is a moon which is not a sun. These results should be compared with the fact that in $C[0, 1]$ every moon is a sun.

Now we prove:

THEOREM 2.8. *Let $V \subset X$. Consider the following statements:*

(1) *V is a sun.*

(2) *For each $x \in X$, every local minimum of the function*

$$\Phi_x(v) = \| x - v \|$$

on V is a global minimum.

(3) *V is a moon.*

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3). *Hence in a* MS-*space, all three statements are equivalent.*

Proof. (1) $\Rightarrow$ (2): Let $x \in X$ and let v_0 be a local minimum for Φ_x, i.e., there exists $\epsilon > 0$ such that $\| x - v_0 \| \leqslant \| x - v \|$ for every $v \in V \cap B(v_0, \epsilon)$. If v_0 is not a global minimum for Φ_x, there exists some $v \in V$ such that $\| x - v \| < \| x - v_0 \|$. Thus $v \in B(x, \| x - v_0 \|) \subset K(v_0, x)$ so $K(v_0, x) \cap V \neq \varnothing$. Let $x_1 = v_0 + \lambda(x - v_0)$ for some $0 < \lambda < \epsilon/2 \| x - v_0 \|$. Then $x_1 \in P_V^{-1}(v_0)$ and $K(v_0, x_1) = K(v_0, x)$. Since V is a sun, $K(v_0, x_1) \cap V = \varnothing$, which is a contradiction.

(2) $\Rightarrow$ (3): If V is not a moon, there exists $v_0 \in V$ and $x \in P_V^{-1}(v_0)$ with $K(v_0, x) \cap V \neq \varnothing$ such that $v_0 \notin \overline{(Kv_0, x) \cap V}$ i.e., there exists $\epsilon > 0$ such that

$$B(v_0, \epsilon) \cap K(v_0, x) \subset X \backslash V.$$

Let $u \in K(v_0, x) \cap V$. Then, for some $\lambda > 0$, $u \in B(v_0 + \lambda(x - v_0), \lambda \| x - v_0 \|)$. Setting $x_1 = v_0 + \lambda(x - v_0)$ we get that $u \in B(x_1, \| x_1 - v_0 \|)$, $K(v_0, x_1) = K(v_0, x)$, and

$$B(v_0, \epsilon) \cap B(x_1, \| x_1 - v_0 \|) \subset X \backslash V,$$

i.e., x_1 has v_0 as a local best approximation in V. But $\| x_1 - u \| < \| x_1 - v_0 \|$ so v_0 is not a global best approximation to x_1.

Remark. In general, none of the implications of Theorem 2.8 is reversible. To see that (2) $\Rightarrow$ (1) is not true, we need only consider the set V which is the complement of the open unit ball in the Euclidean plane. V is obviously not a sun but it is easy to verify that the functions Φ_x have only global minima.

To see that (3) $\Rightarrow$ (2) is generally false, let X be the Euclidean plane and let

$$V = \{(\xi, \eta): \tfrac{1}{4}\xi^2 + \eta^2 \geqslant 1\}.$$

This set is readily seen to be a moon, but the point $x = (0, -\frac{1}{2})$ has $(0, 1)$ as a local best approximation in V which is not a global best approximation.

For the remainder of this section, we investigate the connection between suns and moons with certain continuity properties of the metric projection. We consider a more general concept than the usual concept of lower semicontinuity.

DEFINITION. Let $V \subset X$ and $x_0 \in X$. We say that P_V is outer radially lower continuous (abbreviated ORL-*continuous*) at x_0 if, for each $v_0 \in P_V(x_0)$ and each open set W such that $P_V(x_0) \cap W \neq \varnothing$, there exists a neighborhood U of x_0 such that $P_V(x) \cap W \neq \varnothing$ for every $x \in U \cap \{v_0 + \lambda(x_0 - v_0): \lambda \geqslant 1\}$. P_V is called ORL-continuous if it is ORL-continuous at each point of X. (Note that P_V is obviously ORL-continuous at each point of V.)

LEMMA 2.9. *Let $V \subset X$ and $x_0 \in X$. The following statements are equivalent*:

(1) *P_V is* ORL-*continuous at x_0.*

(2) *For each v_0, v_1 in $P_V(x_0)$ and each $\epsilon > 0$, there exists $\delta > 0$ such that $P_V(x) \cap B(v_1, \epsilon) \neq \varnothing$ for all x in $\{v_0 + \lambda(x_0 - v_0): 1 \leqslant \lambda < 1 + \delta\}$.*

(3) *For each v_0, v_1 in $P_V(x_0)$ and every sequence x_n in $\{v_0 + \lambda(x_0 - v_0): \lambda \geqslant 1\}$ with $x_n \to x_0$, there exists $v_n \in P_V(x_n)$ such that $v_n \to v_1$.*

Proof. (1) $\Rightarrow$ (2) is clear.

(2) $\Rightarrow$ (3): Let v_0, v_1 in $P_V(x_0)$ and $x_n = v_0 + \lambda_n(x_0 - v_0)$ with $\lambda_n > 1$ and $\lambda_n \to 1$ (i.e., $x_n \to x_0$). For $\epsilon = 1$ there exists $\delta_1 > 0$ such that $P_V(x) \cap B(v_1, 1) \neq \varnothing$ for every $x \in V_{\delta_1}$, where $V_\delta = \{v_0 + \lambda(x_0 - v_0): 1 < \lambda < 1 + \delta\}$. Choose n_1 such that $x_n \in V_{\delta_2}$ for every $n \geqslant n_1$. Choose $y_1 \in P_V(x_{n_2}) \cap B(v_1, 1)$. For $\epsilon = \frac{1}{2}$ there exists a δ_2, $0 < \delta_2 < \delta_1$, such that $P_V(x) \cap B(v_1, \frac{1}{2}) \neq \varnothing$ for every $x \in V_{\delta_2}$. Let $n_2 > n_1$ be such that $x_n \in V_{\delta_2}$ for every $n \geqslant n_2$ and choose $y_2 \in P_V(x_{n_2}) \cap B(v_1, \frac{1}{2})$. Continuing in this fashion, we obtain a sequence of integers (n_k), a decreasing sequence of positive numbers (δ_k), and a sequence (y_k) such that $\delta_k \to 0$, $x_n \in V_{\delta_k}$ for every $n \geqslant n_k$, and $y_k \in P_V(x_{n_k}) \cap B(v_1, 1/k)$. We define a sequence (v_n) by taking $v_n \subset P(x_n)$ for $n = 1, \ldots, n_1 - 1$, $v_{n_k} = y_k$ for every k, and $v_n \in P_V(x_n) \cap B(v_1, 1/k)$ for $n_k < n < n_{k+1}$. Then $v_n \in P_V(x_n)$ for every n and $v_n \to v_1$.

(3) $\Rightarrow$ (1): Suppose (3) holds but (1) fails. Then there exists $v_0 \in P_V(x_0)$ and an open set W with $P_V(x_0) \cap W \neq \varnothing$ such that for every neighborhood U of x_0 there exists an x in $U \cap \{v_0 + \lambda(x_0 - v_0): \lambda > 1\}$ such that $P_V(x) \cap W = \varnothing$. Choose $v_1 \in P_V(x_0) \cap W$. Then for every n there exists $x_n = v_0 + \lambda_n(x_0 - v_0)$ with $1 < \lambda_n < 1/n$ such that $P_V(x_n) \cap W = \varnothing$. Then $x_n \to x_0$, but, if $v_n \in P_V(x_n)$, then $v_n \notin W$ so $v_n \not\to v_1$.

THEOREM 2.10. *If V is a sun, then P_V is* ORL-*continuous.*

Proof. Let $x_0 \in X$, v_0, v_1 in $P_V(x_0)$, and $\epsilon > 0$. It suffices to show that if $x = v_0 + \lambda(x_0 - v_0)$, $\lambda > 1$, then $P_V(x) \cap B(v_1, \epsilon) \neq \varnothing$.

Now $v_0 \in P_V(x)$ since V is a sun, and

$$\begin{aligned} \| x - v_1 \| &\leqslant \|(1 - \lambda)(v_0 - x)\| + \| x_0 - v_1 \| \\ &= (\lambda - 1) \| v_0 - x_0 \| + \| x_0 - v_0 \| \\ &= \lambda \| x_0 - v_0 \| = \| x - v_0 \| \leqslant \| x - v_1 \| \end{aligned}$$

implies $\| x - v_1 \| = \| x - v_0 \|$ and $v_1 \in P_V(x)$. This completes the proof.

THEOREM 2.11. *Let $V \subset X$. If P_V is* ORL-*continuous, then "local best approximations are global," i.e., statement* (2) *of Theorem* 2.8 *holds.*

Proof. If not, there exists $x_0 \in X$, $v_0 \in V$, and $\epsilon > 0$ such that $\| x_0 - v_0 \| \leqslant \| x_0 - v \|$ for every $v \in B(v_0, \epsilon) \cap V$, but $\| x_0 - v_0 \| > \operatorname{dist}(x_0, V)$. Let x_1 be the last point on the line segment $[v_0, x_0]$ which has v_0 as a best (global) approximation in V. Thus $\operatorname{dist}(x, V) < \| x - v_0 \|$ for every $x \in (x_1, x_0]$. Choose $\delta > 0$ such that $P_V(x) \cap B(v_0, \epsilon) \neq \varnothing$ for every $x \in V_\delta = \{x_1 + \lambda(x_1 - v_0): 1 < \lambda < 1 + \delta\}$. If $x_\lambda \in V_\delta$ and $v \in P_V(x_\lambda) \cap B(v_0, \epsilon)$, then $\| x_\lambda - v \| < \| x_\lambda - v_0 \|$ and so

$$\| x_0 - v_0 \| = \| x_0 - x_\lambda \| + \| x_\lambda - v_0 \| > \| x_0 - x_\lambda \| + \| x_\lambda - v \| \geqslant \| x_0 - v \|,$$

which contradicts the fact that v_0 is a local best approximation to x_0.

By combining Theorems 2.8, 2.10, and 2.11 we obtain the theorem stated in the introduction. In particular, we have:

THEOREM 2.12. *Let X be* MS-*space. A subset of X is a sun if and only if its metric projection is* ORL-*continuous.*

3. THE SPACE $C_0(T, X)$

Let T be a locally compact Hausdorff space and X a (real) normed linear space. We denote by $C_0(T, X)$ the space of all continuous functions $z: T \to X$ which vanish at infinity, and endowed with the supremum norm:

$$\| z \| = \sup_{t \in T} \| z(t) \|_X .$$

Thus $z \in C_0(T, X)$ if and only if z is a continuous X-valued function on T such that the set $\{t \in T: \| z(t) \|_X \geqslant \epsilon\}$ is compact for every $\epsilon > 0$. With the pointwise linear operations, $C_0(T, X)$ is a normed linear space. Whenever it is necessary to distinguish between the norms in $C_0(T, X)$ and X, we denote the latter by $\|\cdot\|_X$. We often write $C_0(T)$ for $C_0(T, \mathbb{R})$, when $\mathbb{R}$ is the set of real numbers.

For a given $z \in C_0(T, X)$, its critical point set is defined by

$$\operatorname{crit} z = \{t \in T: \| z(t)\|_X = \| z \|\}.$$

A space X is called an (A_c)-*space* if it is an (A)-space and $\mathscr{E}(x)$ is weak*-closed for every nonzero x in X. (The latter condition is always satisfied, in particular, when $\mathscr{E}(X^*)$ is weak*-closed.) The main result of this section is:

THEOREM 3.1. *Let X be an* (A_c)-*space. Then $C_0(T, X)$ is an* (A)-*space.*

COROLLARY 3.2. *If X is a finite-dimensional polyhedral space, then $C_0(T, X)$ is an* (A)-*space. In particular, $C_0(T)$ is an* (A)-*space.*

The proof of Theorem 3.1 depends on a number of lemmas which are of independent interest. If $x^* \in X^*$ and $t \in T$, we denote by $z^* = x^*(\cdot(t))$ the element of $C_0(T, X)^*$ defined by $z^*(z) = x^*(z(t))$ for every $z \in C_0(T, X)$. Our first lemma characterizes the extreme points of the unit ball in $C_0(T, X)^*$ and was proved in collaboration with P. D. Morris. It generalizes a result of Singer [13], who proved it in the case in which T is compact and X a Banach space. Our proof, as distinct from his, is independent of the representation of the elements of the dual of $C_0(T, X)$.

LEMMA 3.3. *Let X be a* (*real or complex*) *normed linear space and $Z = C_0(T, X)$. Then*

$$\mathscr{E}(Z^*) = \{x^*(\cdot(t)): x^* \in \mathscr{E}(X^*), t \in T\}.$$

Proof. Let $A = \{x^*(\cdot(t)): x^* \in B(X^*), t \in T\}$.

CLAIM. A is a weak*-closed (hence compact) subset of $B(Z^*)$. For let $z_\alpha^* = x_\alpha^*(\cdot(t_\alpha))$ be a net in A which converges weak* to some $z^* \in B(Z^*)$.

Case 1. (t_α) has a cluster point $t \in T$. Then there is a subnet (t_β) such that $t_\beta \to t$. Now (z_β^*) also converges weak* to z^*. By passing to a further subnet of (x_β^*) if necessary, we may assume that (x_β^*) converges weak* to some $x^* \in B(X^*)$. Thus, for every $z \in Z$,

$$z^*(z) = \lim_\beta z_\beta^*(z) = \lim_\beta x_\beta^*(z(t_\beta)) = x^*(z(t))$$

and hence $z^* = x^*(\cdot(t)) \in A$.

Case 2. (t_α) has no cluster point. If $z \in Z$ and $\epsilon > 0$, choose a compact set $T_0 \subset T$ such that $\| z(t)\| < \epsilon$ for every $t \in T \backslash T_0$. Since (t_α) has no cluster point, there exists an index α_0 such that $t_\alpha \in T\backslash T_0$ for every $\alpha \geqslant \alpha_0$. Hence, for every $\alpha \geqslant \alpha_0$,

$$| z_\alpha^*(z)| = | x_\alpha^*(z(t_\alpha))| \leqslant \| z(t_\alpha)\| < \epsilon$$

implies that z_α^* converges weak* to $0 \in A$.

CLAIM. The weak*-closed convex hull of A, $\overline{\text{co}}(A)$, is equal to $B(Z^*)$. For, if not, then by a well-known separation theorem [9, p. 417] there would exist $z \in Z$, and $z_0^* \in B(Z^*)\backslash\overline{\text{co}}(A)$, such that

$$\begin{aligned} \operatorname{Re} z_0^*(z) &> \sup\{\operatorname{Re} z^*(z): z^* \in \overline{\text{co}}(A)\} \\ &\geqslant \sup\{\operatorname{Re} x^*(z(t)): x^* \in B(X^*),\ t \in T\} \\ &= \| z \|, \end{aligned}$$

which is absurd.

CLAIM. $\mathscr{E}(Z^*) \subset \{x^*(\cdot(t)): x^* \in \mathscr{E}(X^*),\ t \in T\}$.

Let $E = \{x^*(\cdot(t)): x^* \in \mathscr{E}(X^*),\ t \in T\}$. By a theorem of Milman (cf., e.g., [9, p, 440]), $\mathscr{E}(Z^*) \subset A$. If there is some $z^* \in \mathscr{E}(Z^*)\backslash E$, then $z^* = x^*(\cdot(t))$ for some $x^* \in B(X^*)\backslash\mathscr{E}(X^*)$ and some $t \in T$. Hence there exist x_1^*, x_2^* in $B(X^*)$, $x_1^* \neq x_2^*$ such that $x^* = \frac{1}{2}(x_1^* + x_2^*)$ and so $z^* = \frac{1}{2}[x_1^*(\cdot(t)) + x_2^*(\cdot(t))]$, which contradicts $z^* \in \mathscr{E}(Z^*)$. Thus $\mathscr{E}(Z^*) \subset E$.

To complete the proof of Lemma 3.3, we must show that $E \subset \mathscr{E}(Z^*)$. Let $x_0^* \in \mathscr{E}(X^*)$, $t_0 \in T$, and suppose

$$x_0^*(\cdot(t_0)) = \tfrac{1}{2}[z_1^* + z_2^*] \qquad \text{for some} \quad z_i^* \in B(Z^*).$$

Let $z \in Z$, $\| z \| \leqslant 1$, and suppose $x_0^*(z(t_0)) = 0$. We will show that $z_1^*(z) = z_2^*(z) = 0$. Fix an arbitrary $\epsilon > 0$. Let $T_0 = \{t: \| z(t)\| \geqslant \epsilon\}$ and $Y = \{t: \| z(t)\| > \epsilon/2\}$. Then U is a neighborhood of the compact set T_0. By Urysohn's lemma, we choose $f \in C_0(T)$ such that $0 \leqslant f \leqslant 1, f = 1$ on T_0, and $f = 0$ off U. Set $z_1 = zf$. Then $z_1 \in Z$, $\| z_1 \| \leqslant 1$, and, for every $t \in T$,

$$\| z(t) - z_1(t)\| = [1 - f(t)]\, \| z(t)\| < \epsilon,$$

i.e., $\| z - z_1 \| < \epsilon$. Again by Urysohn's lemma, we choose $g \in C_0(T)$, with $0 \leqslant g \leqslant 1$, such that $g(t_0) = 1$ and $g = 0$ off the set $\{t: \| z(t)\| < \epsilon/2\}$. Choose $x \in S(X)$ such that $\operatorname{Re} x_0^*(x) > 1 - (\epsilon/4)$ and set $z_2 = xg$. Then $z_2 \in Z$, $\| z_2 \| = 1$, and $z_2(t_0) = x$ so

$$1 - \frac{\epsilon}{4} < \operatorname{Re} x_0^*(z_2(t_0)) = \frac{1}{2}[\operatorname{Re} z_1^*(z_2) + \operatorname{Re} z_2^*(z_2)],$$

which implies

$$\min\{\operatorname{Re} z_1^*(z_2), \operatorname{Re} z_2^*(z_2)\} > 1 - (\epsilon/2). \tag{1}$$

Also, $\| z_1 + z_2 \| = 1$ implies that

$$1 - \frac{\epsilon}{4} < \operatorname{Re} x_0^*(z_2(t_0)) = \operatorname{Re} x_0^*[z_1(t_0) + z_2(t_0)]$$
$$= \frac{1}{2} [\operatorname{Re} z_1^*(z_1 + z_2) + \operatorname{Re} z_2^*(z_1 + z_2)],$$

from which it follows that

$$\min\{\operatorname{Re} z_1^*(z_1 + z_2), \operatorname{Re} z_2^*(z_1 + z_2)\} > 1 - (\epsilon/2). \tag{2}$$

Combining (1) and (2) we get

$$[\operatorname{Re} z_i^*(z_1)] < \epsilon/2 \qquad \text{for} \quad i = 1, 2.$$

A routine computation shows that

$$| \operatorname{Im} z_i^*(z_2)| \leqslant \sqrt{\epsilon}, \qquad | \operatorname{Im} z_i^*(z_1 + z_2)| \leqslant \sqrt{\epsilon},$$

for $i = 1, 2$. Thus

$$| \operatorname{Im} z_i^*(z_1)| \leqslant | \operatorname{Im} z_i^*(z_1 + z_2)| + | \operatorname{Im} z_i^*(z_2)| \leqslant 2 \sqrt{\epsilon}$$

for $i = 1, 2$. Hence, for $i = 1, 2$,

$$| z_i^*(z)| \leqslant | z_i^*(z - z_1)| + | z_i^*(z_1)|$$
$$\leqslant \| z - z_1 \| + \sqrt{(\epsilon^2/4) + 4\epsilon} < \epsilon + \sqrt{(\epsilon^2/4) + 4\epsilon}.$$

Since ϵ was arbitrary, this shows that $z_1^*(z) = z_2^*(z) = 0$. We have shown that $\| z \| \leqslant 1$ and $x_0^*(z(t_0)) = 0$ imply $z_1^*(z) = z_2^*(z) = 0$. It follows that $z_i^* = \alpha_i x_0^*(\cdot(t))$ for some scalars α_i $(i = 1, 2)$. Since $x_0^*(\cdot(t_0)) = \frac{1}{2}(z_1^* + z_2^*)$, $\alpha_1 = \alpha_2 = 1$ and $x_0^*(\cdot(t_0)) \in \mathscr{E}(Z^*)$. This completes the proof.

It is well known [12] that, if Y is a subspace of the normed linear space Z, then $\mathscr{E}(Y^*) \subset \{z^* |_Y : z^* \in \mathscr{E}(Z^*)\}$. Thus

Corollary 3.4. *Let Y be a subspace of $C_0(T, X)$. Then*

$$\mathscr{E}(Y^*) \subset \{x^*(\cdot(t)) : x^* \in \mathscr{E}(X^*),\ t \in T\}.$$

We state the following simple result for reference purposes since it is used several times:

LEMMA 3.5. *Let X and Z be as in Lemma 3.3. Then for each compact set $T_0 \subset T$, each neighborhood U of T_0, and each $x \in X$, there exists $z \in Z$ such that $z = x$ on T_0, $z = 0$ off U, and $\|z\| = \|x\|$.*

Proof. By Urysohn's lemma there exists $f \in C_0(T)$ such that $0 \leqslant f \leqslant 1$, $f = 1$ on T_0, and $f = 0$ off U. The element $z = xf$ works.

LEMMA 3.6. *Let X and Z be as in Lemma 3.3. Then $\mathscr{E}(Z^*)$ is homeomorphic to the product $T \times \mathscr{E}(X^*)$ (relative to the σ-topologies on $\mathscr{E}(Z^*)$ and $\mathscr{E}(X^*)$).*

Proof. By Lemma 3.3,

$$\mathscr{E}(Z^*) = \{x^*(\cdot(t)) : x^* \in \mathscr{E}(X^*), t \in T\}.$$

We define $F: T \times \mathscr{E}(X^*) \to \mathscr{E}(Z^*)$ by

$$F[(t, x^*)] = x^*(\cdot(t)).$$

F is clearly onto and continuous. Suppose $x_1^*(\cdot(t)) = x_2^*(\cdot(t))$. If $t_1 \neq t_2$, then taking an $x \in X \backslash X_1^{*-1}(0)$, an application of Lemma 3.5 yields a $z \in Z$ such that $z(t_1) = x$ and $z(t_2) = 0$. But this implies the contradiction $x_1^*(z(t_1)) \neq x_2^*(z(t_2))$. Hence $t_1 = t_2$. A similar argument shows $x_1^* = x_2^*$. Thus F is one-to-one. To show F^{-1} is continuous, let $x_\alpha^*(\cdot(t_\alpha)) \to x^*(\cdot(t))$. We must verify that $x_\alpha^* \to x^*$ and $t_\alpha^* \to t$. If $t_\alpha^* \not\to t$, there exists a neighborhood U of t and a subnet (t_β) such that $t_\beta \notin U$ for every β. By Lemma 3.5 there is $z \in Z$ such that $z(t) = x \in X \backslash x^{*-1}(0)$ and $z = 0$ off U. Then

$$0 \neq x^*(x) = x^*(z(t)) = \lim x_\beta^*(z(t_\beta)) = 0,$$

which is absurd. Thus $t_\alpha \to t$. Now let $x \in X$ be arbitrary. Choose a neighborhood U of t such that $\bar{U}$ is compact, and choose α_0 such that $t_\alpha \in U$ for every $\alpha \geqslant \alpha_0$. By Lemma 3.5 there exists $z \in Z$ such that $z = x$ on U. Then, for every $\alpha \geqslant \alpha_0$,

$$x_\alpha^*(x) = x_\alpha^*(z(t_\alpha)) \to x^*(z(t)) = x^*(x).$$

Hence $x_\alpha^* \to x^*$ and the proof is complete.

DEFINITION. Let X and Y be topological spaces. A set-valued map $\Psi: X \to 2^Y$ (the set of all closed subsets of Y) is said to be *upper semicontinuous* (abbreviated u.s.c.) at x_0 if for each neighborhood W of $\Psi(x_0)$ there exists a neighborhood U of x_0 such that $\Psi(x) \subset W$ for every $x \subset U$. Ψ is called u.s.c. if it is u.s.c. at each point.

LEMMA 3.7. *Let X be an* (A)*-space. Then (relative to the σ-topology on $\mathscr{E}(X^*)$) the peak set mapping $x \to \mathscr{E}(x)$ is u.s.c.*

Proof. Let $x_0 \in X$ and W be a σ-open neighborhood of $\mathscr{E}(x_0)$. If $x_0 = 0$, the result is trivially true. If $x_0 \neq 0$, there is a $y \in X$ such that $\mathscr{E}(y) \subset W$ and

$$\sup\{x^*(x_0): x^* \in \mathscr{E}(X^*)\backslash\mathscr{E}(y)\} = \| x_0 \| - \delta$$

for some $\delta > 0$. Let

$$U = \{x \in X: \| x - x_0 \| < \delta/2\}.$$

If $x \in U$ and $x^* \in \mathscr{E}(X^*)\backslash\mathscr{E}(y)$, we have

$$\begin{aligned} x^*(x) &= x^*(x - x_0) + x^*(x_0) \\ &\leqslant \| x - x_0 \| + \| x_0 \| - \delta < \| x_0 \| - \delta/2 < \| x \|, \end{aligned}$$

so $x^* \notin \mathscr{E}(x)$. Thus $\mathscr{E}(x) \subset \mathscr{E}(y) \subset W$ for each $x \in U$.

Proof of Theorem 3.1. Let $z \in S(Z)$ and let W be a σ-open neighborhood of $\mathscr{E}(z)$. We have to show the existence of a $y \in Z$ such that $W \supset \mathscr{E}(y)$ and

$$\sup\{z^*(z): z^* \in \mathscr{E}(Z^*)\backslash\mathscr{E}(y)\} < 1.$$

By Lemma 3.6, we may assume that

$$W = \bigcup_{i \in I} F(A_i \times U_i) \supset \mathscr{E}(z),$$

where F is the homeomorphism constructed in Lemma 3.6, I is some index set, and A_i and U_i are open in T and $\mathscr{E}(X^*)$, respectively. By assumption, $\mathscr{E}(z(t))$ is compact whenever $z(t) \neq 0$. Using Lemma 3.7, we deduce that the map $t \to \mathscr{E}(z(t))$, and hence also the map $t \to \{t\} \times \mathscr{E}(z(t))$, is u.s.c. Since crit z is compact, it follows by a theorem of Michael [11] that

$$\bigcup_{t \in \operatorname{crit} z} [\{t\} \times \mathscr{E}(z(t))],$$

and hence

$$\mathscr{E}(z) = F\Big(\bigcup_{t \in \operatorname{crit} z} [\{t\} \times \mathscr{E}(z(t))]\Big),$$

is compact. Thus there exist sets $A_1 \times U_1, \ldots, A_\nu \times U_\nu$ such that

$$\mathscr{E}(z) \subset \bigcup_{i=1}^{\nu} F(A_i \times U_i) \subset W.$$

For each $t \in \operatorname{crit} z$, define

$$\begin{aligned} a(t) &= \{i \in N: A_i \times U_i \cap \{t\} \times \mathscr{E}(z(t)) \neq \varnothing\} \neq \varnothing, \\ \alpha(t) &= \bigcup_{i \in a(t)} U_i \supset \mathscr{E}(z(t)), \end{aligned}$$

and

$$A(t) = \bigcap_{i \in a(t)} A_i \supset \{t\}.$$

Note that $A(t)$ is open and

$$\mathscr{E}(z) \subset \bigcup_{t \in \operatorname{crit} z} F[\{t\} \times \alpha(t)] \subset \bigcup_{t \in \operatorname{crit} z} F[A(t) \times \alpha(t)] \subset W.$$

Now $t \to \mathscr{E}(z(t))$ is u.s.c. and so, for each $t \in \operatorname{crit} z$, there exists a neighborhood $B(t)$ of t such that $B(t) \subset A(t)$ and $\mathscr{E}(z(t')) \subset \alpha(t)$ for every $t' \in B(t)$. Thus

$$\mathscr{E}(z) \subset \bigcup_{t \in \operatorname{crit} z} F[B(t) \times \alpha(t)] \subset W.$$

Let $C = \bigcup_{t \in \operatorname{crit} z} B(t)$. Then, for each $t' \in C$, we see that

$$F[\{t'\} \times \mathscr{E}(z(t'))] \subset \bigcup_{t \in \operatorname{crit} z} F[B(t) \times \alpha(t)] \subset W.$$

We now choose an open set M such that

$$\operatorname{crit} z \subset M \subset \overline{M} \subset \bigcup_{t \in \operatorname{crit} z} B(t),$$

$\overline{M}$ is a compact G_δ, and

$$M \subset \{t : \| z(t) \| > \tfrac{1}{2}\}.$$

Next we extend the definition of $\alpha(t)$ and $B(t)$ to all of $\overline{M}$. Since $\overline{M}$ is compact, there exist $t_1, \ldots, t_n$ in $\operatorname{crit} z$ such that $\overline{M} \subset \bigcup_1^n B_k$, where $B_k = B(t_k)$. For each $t \in \overline{M} \backslash \operatorname{crit} z$, define

$$B(t) = \bigcap \{B_k : t \in B_k\}$$

and

$$\alpha(t) = \bigcup \{\alpha(t_k) : t \in B_k\}.$$

Note that $B(t)$ is a neighborhood of t and $\alpha(t)$ is open.

Let $t \in \overline{M} \backslash \operatorname{crit} z$. Then, for each $t' \in B(t)$, $\mathscr{E}(z(t')) \subset \alpha(t)$. Also,

$$F[\{t\} \times \mathscr{E}(z(t))] \subset \bigcup_{\bar{t} \in \overline{M}} F[B(\bar{t}) \times \alpha(\bar{t})] \subset W.$$

To verify the relation $\bigcup_{\bar{t} \in \overline{M}} F[B(\bar{t}) \times \alpha(\bar{t})] \subset W$, it suffices to show $F[B(\bar{t}) \times \alpha(\bar{t})] \subset W$ for every $\bar{t} \in \overline{M} \backslash \operatorname{crit} z$. Now

$$B(\bar{t}) \times \alpha(\bar{t}) = \bigcap \{B_k : \bar{t} \in B_k\} \times \bigcap \{\alpha(t_k) : \bar{t} \in B_k\}$$

$$= \bigcup_{\{k : \bar{t} \in B_k\}} \left\{ \bigcap \{B_j : \bar{t} \in B_j\} \times \alpha(t_k) \right\} \subset \bigcup_{\{k : \bar{t} \in B_k\}} B_k \times \alpha(t_k)$$

implies

$$F[B(\bar{t}) \times \alpha(\bar{t})] \subset \bigcup_{\{k:\bar{t}\in B_k\}} F[B_k \times \alpha(t_k)] \subset W.$$

Since X has the (A)-property, for each $t \in \overline{M}$ there exists $x_t \in X$, $\| x_t \| = 1$, such that

$$\mathring{\mathscr{E}}(x_t) \subset \mathscr{E}(x_t) \subset \alpha(t)$$

and

$$\sup\{x^*(z(t)) \colon x^* \in \mathscr{E}(X^*)\backslash\mathscr{E}(x_t)\} = \| z(t)\| - \epsilon_t$$

for some $\epsilon_t > 0$. By the continuity of z and the u.s.c. of $t \to \mathscr{E}(z(t))$, for each $t \in \overline{M}$ there is a neighborhood $D(t)$ of t such that $D(t) \subset B(t)$ and, for each $t' \in D(t)$,

$$\mathscr{E}(z(t')) \subset \mathring{\mathscr{E}}(x_t), \qquad \text{and} \qquad \| z(t') - z(t)\| < \epsilon_t/2.$$

Since $\overline{M}$ is compact and $\bigcup_{t\in\overline{M}} D(t) \supset \overline{M}$ there exist $s_1, \ldots, s_m$ in $\overline{M}$ such that $\overline{M} \subset \bigcup_{\mu=1}^{m} D_\mu$, where $D_\mu = D(S_\mu)$.

Since $\overline{M}$ is a compact G_δ, we can choose a "partition of unity" $\rho_1, \ldots, \rho_m$ in $C_0(T)$ as follows: $\rho_i \geqslant 0$ for every i, $\rho_i = 0$ on $\overline{M}\backslash D_i$, $\sum_i^m \rho_i = 1$ on $\overline{M}$, and $\sum_1^m \rho_i < 1$ off $\overline{M}$. We set

$$y(t) = \sum_1^m \rho_i(t)\, x_i, \qquad \text{where} \quad x_i = x_{s_i}.$$

For each $t \in \overline{M}$, put

$$b(t) = \{i \in N \colon \rho_i(t) \neq 0\}.$$

We now show that

$$\mathscr{E}(y\; t)) = \bigcap_{i\in b(t)} \mathscr{E}(x_i) \qquad \text{for every} \quad t \in \overline{M}.$$

First note that $\bigcap_{i\in b(t)} \mathscr{E}(x_i) \neq \varnothing$ since $\mathscr{E}(x_i) \supset \mathscr{E}(z(t))$ for $i \in b(t)$. Now $x^* \in \bigcap_{i\in b(t)} \mathscr{E}(x_i)$ implies

$$x^*(y(t)) = \sum_1^m \rho_i(t)\, x^*(x_i) = \sum_{i\in b(t)} \rho_i(t) = 1 \geqslant \| y(t)\|.$$

Thus $x^* \in \mathscr{E}(y(t))$. Conversely, if $x^* \in \mathscr{E}(y(t))$, then for $i_0 \in b(t)$

$$1 = \| y(t)\| = x^*(y(t)) = \sum \rho_i(t)\, x^*(x_i) < \rho_{i_0}(t)\, x^*(x_{i_0}) + 1 - \rho_{i_0}(t),$$

which implies that $1 \leqslant x^*(x_{i_0})$ and so $x^* \in \mathscr{E}(x_{i_0})$.

We have $|y(t)| = \|y\| = 1$ for every $t \in \overline{M}$ and crit $y \supset \overline{M}$. But $\sum \rho_i < 1$ off $\overline{M}$ so $|y(t)| < 1$ off $\overline{M}$ and crit $y = \overline{M}$. For every $t \in \overline{M}$,

$$\bigcap_{i \in b(t)} \mathscr{E}(x_i) = \mathscr{E}(y(t)) \subset \bigcap_{\mu \in b(t)} \alpha(s_\mu),$$

so

$$\{t\} \times \mathscr{E}(y(t)) \subset \bigcap_{i \in b(t)} D_i \times \bigcap_{\mu \in b(t)} \alpha(s_\mu) \subset D_i \times \alpha(s_i) \subset B(s_i) \times \alpha(s_i)$$

for every $i \in b(t)$. Hence

$$F[\{t\} \times \mathscr{E}(y(t))] \subset F[B(s_i) \times \alpha(s_i)] \subset W.$$

But

$$\mathscr{E}(y) = \bigcup_{t \in \overline{M}} F[\{t\} \times \mathscr{E}(y(t))]$$

and so $\mathscr{E}(y) \subset W$.

It remains to show that

$$\sup\{z^*(z): z^* \in \mathscr{E}(Z^*)\backslash\mathscr{E}(y)\} < 1,$$

i.e.,

$$\sup\{x^*(z(t)): x^*(\cdot(t)) \in \mathscr{E}(Z^*)\backslash\mathscr{E}(y)\} < 1.$$

Now $x^*(\cdot(t)) \in \mathscr{E}(Z^*)\backslash\mathscr{E}(y)$ if and only if $x^* \in \mathscr{E}(X^*)$ and either $t \notin$ crit $y = \overline{M}$ or $x^* \notin \mathscr{E}(y(t))$, $t \in$ crit y. If $t \in \overline{M}$ and $x^* \notin \mathscr{E}(y(t))$, then $x^* \notin \mathscr{E}(x_i)$ for some $i \in b(t)$ implies $t \in D_i$ and

$$\begin{aligned} x^*(z(t)) &= x^*(z(t) - z(t_i)) + x^*(z(t_i)) \\ &< \tfrac{1}{2}\epsilon_{t_i} + \| z(t_i)\| - \epsilon_{t_i} = \| z(t_i)\| - \tfrac{1}{2}\epsilon_{t_i} \\ &\leqslant 1 - \epsilon_0, \end{aligned}$$

where $\epsilon_0 = \frac{1}{2} \min_{1 \leqslant i \leqslant n} \epsilon_{t_i}$. Also,

$$\sup\{\| z(t)\|: t \notin \overline{M}\} = 1 - \epsilon_1$$

for some $\epsilon_1 > 0$. Taking $\epsilon = \min\{\epsilon_0, \epsilon_1\}$, we conclude

$$\sup\{x^*(z(t)): x^*(\cdot(t)) \in \mathscr{E}(Z^*)\backslash\mathscr{E}(y)\} \leqslant 1 - \epsilon < 1,$$

and the proof is complete.

4. Product Spaces

Let (X_i), $i \in I$, be a collection of real normed linear spaces. By the l_∞-product of the X_i, denoted $(\prod_{i\in I} X_i)_{l_\infty}$, we mean the set of all functions x on I such that $x(i) \in X_i$ for each i (i.e., $x \in \prod_{i\in I} X_i$) and $\sup_{i\in I} \| x(i)\| < \infty$. Defining, the linear operations pointwise and a norm by $\| x \| = \sup_{i\in I} \| x(i)\|$, $(\prod_{i\in I} X_i)_{l_\infty}$ is a normed linear space. By the c_0-product of the X_i, denoted $(\prod_{i\in I} X_i)_{c_0}$, we mean the subspace of $(\prod_{i\in I} X_i)_{l_\infty}$ consisting of those x such that the set $\{i \in I: \| x(i)\| \geqslant \epsilon\}$ is finite for every $\epsilon > 0$. Similarly, the l_1-product of the X_i, denoted $(\prod_{i\in I} X_i)_{l_1}$, is the set of all functions x in $\prod_{i\in I} X_i$ such that the norm $\| x \| = \sum_{i\in I} \| x(i)\|$ is finite.

It is well known (cf. [8]) that $(\prod_i X_i^*)_{l_1}$ (respectively, $(\prod_i X_i^*)_{l_\infty}$) is isometric to $(\prod_i X_i)_{c_0}$ (respectively, $(\prod_i X_i)_{l_1}^*$) via the mapping $(x^*(i))_{i\in I} \to x^*$ defined by

$$x^*(x) = \sum_{i\in I} x^*(i)\, x(i)$$

for every x in the product space. Also, it can be readily verified that if $X = (\prod_i X_i)_{l_1}$ (respectively, $X = (\prod_i X_i)_{l_\infty}$), then

$$\mathscr{E}(X) = \{x \in X: x(\alpha) \in \mathscr{E}(X_\alpha) \text{ for some } \alpha \text{ and } x(i) = 0 \text{ if } i \neq \alpha\}$$

(respectively, $\mathscr{E}(X) = \{x \in X: x(i) \in \mathscr{E}(X_i) \text{ for every } i\}$).

We first consider the c_0-product $X = (\prod_i X_i)_{c_0}$. For any $x \in X$, we define

$$\operatorname{crit} x = \{i \in I: \| x(i)\| = \| x \|\}.$$

THEOREM 4.1. *$(\prod_i X_i)_{c_0}$ is an (A)-space if and only if each X_i is an (A)-space.*

The essential part of the proof is contained in

LEMMA 4.2. *Let $X = (\prod_{i\in I} X_i)_{c_0}$, and $x \in X\backslash\{0\}$. Then*

(a) *$\mathscr{E}(x) = \{x^* \in X: x^*(j) \in \mathscr{E}(x(j))$ for some $j \in \operatorname{crit} x$ and $x^*(i) = 0$ if $i \neq j\}$.*

(b) *If W_α is σ-open in $\mathscr{E}(X_\alpha^*)$ and $W = \prod_{i\in I} W_i$, where $W_i = \{0\}$ if $i \neq \alpha$, then W is σ-open in $\mathscr{E}(X^*)$.*

(c) *If W is σ-open in $\mathscr{E}(X^*)$ and $W \supset \mathscr{E}(x)$, then, for each $j \in \operatorname{crit} x$, the set*

$$W_j = \{x^*(j): x^* \in W\} \cap \mathscr{E}(X_j^*)$$

is σ-open in $\mathscr{E}(X_j^*)$ and $W_j \supset \mathscr{E}(x(j))$.

Proof. The proof of (a) is routine and omitted. For (b), let $x_\alpha^* \in W_\alpha$. Then there exist $x_{\alpha 1}, \ldots, x_{\alpha n}$ in X_α and $\epsilon > 0$ such that the set

$$U_\alpha = \bigcap_{k=1}^{n} \{y_\alpha^* \in \mathscr{E}(X_\alpha^*): | y_\alpha^*(x_{\alpha k}) - x_\alpha^*(x_{\alpha k})| < \epsilon\}$$

is contained in W_α. We can assume $x_\alpha^*(x_{\alpha k}) \neq 0$ for some k. Then, for $\epsilon > 0$ small enough, $y_\alpha^* \in U_\alpha$ implies $y_\alpha^*(x_{\alpha k}) \neq 0$ for some k. Define $x_k \in X$ by putting $x_k(\alpha) = x_{\alpha k}$ and $x_k(i) = 0$ if $i \neq \alpha$. Let $x^* \in X^*$ be defined by $x^*(\alpha) = x_\alpha^*$ and $x^*(i) = 0$ if $i \neq \alpha$. Then the σ-open neighborhood of x^*

$$U = \bigcap_{k=1}^{n} \{y^* \in \mathscr{E}(X^*): | y^*(x_k) - x^*(x_k)| < \epsilon\}$$

has the property that, if $y^* \in U$, then $y^*(\alpha) \in U_\alpha \subset W_\alpha$ and $y^*(i) = 0$ if $i \neq \alpha$. Thus $U \subset W$ so W is σ-open.

For the proof of (c), we first observe that $W_j \supset \mathscr{E}(x(j))$ for every $j \in \operatorname{crit} x$ is clear. Let $t \in \operatorname{crit} x$ and $x_j^* \in W_j$. Then there exists $x^* \in W$ such that $x^*(j) = x_j^*$ and $x^*(i) = 0$ if $i \neq j$. Choose a σ-open set

$$U = \bigcap_{k=1}^{n} \{y^* \in \mathscr{E}(X^*): | y^*(x_k) - x^*(x_k)| < \epsilon\}$$

so that $W \supset U$. Then we see that

$$W_j \supset \bigcap_{k=1}^{n} \{y^*(j) \in \mathscr{E}(X^*): | y^*(j)\, x_k(j) - x^*(j)\, x_k(j)| < \epsilon\}$$

and the right side is a σ-open neighborhood of $x^*i(j)$. Thus W_j is σ-open.

Proof of Theorem 4.1. Let $X = (\prod X_i)_{c_0}$. Suppose that X is an (A)-space. Fix an arbitrary index α, let $x_\alpha \in S(X_\alpha)$, and let W_α be a σ-open set in $\mathscr{E}(X_\alpha^*)$ such that $W_\alpha \supset \mathscr{E}(x_\alpha)$. Define $x \in X$ by setting $x(i) = 0$ if $i \neq \alpha$ and $x(\alpha) = x_\alpha$. Let $W = \prod_i W_i$ where $W_i = \{0\}$ if $i \neq \alpha$. Then W is σ-open in $\mathscr{E}(X^*)$ by part (b) of Lemma 4.2 and $\mathscr{E}(x) \subset W$. By the (A)-property, there exists a $y \in X$ such that $\mathscr{E}(y) \subset W$ and

$$\sup\{x^*(x): x^* \in \mathscr{E}(X^*) \backslash \mathscr{E}(y)\} < 1.$$

It follows that $\operatorname{crit} y = \operatorname{crit} x = \alpha$, $\mathscr{E}(y_\alpha) \subset W_\alpha$, and

$$\begin{aligned}
&\sup\{x^*(\alpha)\, x(\alpha): x^*(\alpha) \in \mathscr{E}(X_\alpha^*) \backslash \mathscr{E}(y(\alpha))\} \\
&\quad \leqslant \sup\{x^*(x): \text{either } i \in \operatorname{crit} y \text{ and } x^*(i) \in \mathscr{E}(X_i^*) \backslash \mathscr{E}(y(i)) \\
&\qquad\qquad \text{or } i \notin \operatorname{crit} y \text{ and } x^*(i) \in \mathscr{E}(X_i^*)\} \\
&\quad = \sup\{x^*(x): x^* \in \mathscr{E}(X^*) \backslash \mathscr{E}(y)\} < 1.
\end{aligned}$$

Thus X_α is an (A)-space.

Now suppose each X_i is an (A)-space. Let $x \in S(X)$ and W be a σ-open set with $W \supset \mathscr{E}(x)$. By Lemma 4.2(c), for each $j \in \operatorname{crit} x$, the set $W_j = \{x^*(j): x^* \in W\} \cap \mathscr{E}(X_j^*)$ is σ-open in $\mathscr{E}(X_j^*)$ and $W_j \supset \mathscr{E}(x(j))$. By the (A)-property, there exists for each $j \in \operatorname{crit} x$ an element $y_j \in S(X_j)$ such that $\mathscr{E}(y_j) \subset W_j$ and

$$\sup\{x^*(j)\, x(j): x^*(j) \in \mathscr{E}(X_j^*) \backslash \mathscr{E}(y_j)\} = 1 - \epsilon_j$$

for some $\epsilon_j > 0$. Choose $\epsilon_0 > 0$ such that $\operatorname{crit} x = \{j: \| x(j) \| > 1 - \epsilon_0\}$. Define y in X by $y(j) = y_j$ if $j \in \operatorname{crit} x$ and $y(i) = 0$ if $i \notin \operatorname{crit} x$. Then $\operatorname{crit} y = \operatorname{crit} x$, $\mathscr{E}(y) \subset W$, and

$$\begin{aligned} &\sup\{x^*(x): x^* \in \mathscr{E}(X^*) \backslash \mathscr{E}(y)\} \\ &\quad = \sup\{x^*(x): \text{either } i \in \operatorname{crit} y \text{ and } x^*(i) \in \mathscr{E}(X_i^*) \backslash \mathscr{E}(y(i)) \\ &\qquad \text{or } i \notin \operatorname{crit} y \text{ and } x^*(i) \in \mathscr{E}(X_i^*)\} \\ &\quad \leqslant 1 - \min_{j \in \operatorname{crit} x} \{\epsilon_0, \epsilon_j\} < 1. \end{aligned}$$

Thus X is an (A)-space and the proof is complete.

We next consider l_1-products of certain (A)-spaces.

THEOREM 4.3. *Let $(X_i)_{i \in I}$ be a collection of (A)-spaces such that $\mathscr{E}(X_i^*)$ is weak* closed for each i. Then $(\prod_i X_i)_{l_1}$ is an (A)-space.*

Proof. Let $X = (\prod X_i)_{l_1}$. We first note the natural identification of $\mathscr{E}(X^*)$ with $\prod_{i \in I} \mathscr{E}(X_i^*)$. In fact, taking the σ-topologies on $\mathscr{E}(X^*)$ and $\mathscr{E}(X_i^*)$ and the product topology on $\prod_i \mathscr{E}(X_i^*)$, this identification is easily seen to be a homeomorphism (using the fact that convergence in the product topology is equivalent to coordinatewise convergence). In particular, $\mathscr{E}(X^*)$ is weak*-closed (hence σ-compact). Now let $x \in S(X)$ and let W be a σ-open set in $\mathscr{E}(X^*)$ which contains $\mathscr{E}(x)$. By the σ-compactness of $\mathscr{E}(x)$, it follows that there exist a finite number of σ-open sets of the type

$$V_k = \prod_{i \in I} U_i \qquad (k = 1,..., n),$$

where $U_i = \mathscr{E}(X_i^*)$ for all but finitely many indices $i(k, 1),..., i(k, n_k)$ and U_i is σ-open in $\mathscr{E}(X_i^*)$ if $i = i(k, \nu)$ $(\nu = 1,..., n_k)$, such that

$$\mathscr{E}(x) \subset \bigcup_{k=1}^{n} V_k \subset W.$$

Define $I_0 = \{i(k, \nu): k = 1,..., n; \nu = 1,..., n_k\}$ and $\operatorname{supp} x = \{i \in I: x(i) \neq 0\}$. for any $j \in \operatorname{supp} x$ we have

$$\mathscr{E}(x(j)) \subset \operatorname{pr}_j \left(\bigcup_{k=1}^{n} V_k \right),$$

where pr_j denotes the projection onto the jth coordinate space. Since X_j is an (A)-space, there is $y_j \in X_j$ such that

$$\mathscr{E}(y_j) \subset \operatorname{pr}_j \left(\bigcup_{k=1}^{n} V_k \right)$$

and

$$\sup\{x^*(j)\, x(j): x^*(j) \in \mathscr{E}(X_j^*)\backslash\mathscr{E}(y_j)\} = \| x(j)\| - \epsilon_j$$

for some $\epsilon_j > 0$. Next we define the element $y \in X$ by

$$y(i) = \begin{cases} y_i, & \text{if } \ i \in \operatorname{supp} x \cap I_0, \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\mathscr{E}(y) \subset \bigcup_{i}^{n} V_k \subset W.$$

Also, if $x^* \in \mathscr{E}(X^*)\backslash\mathscr{E}(y)$, then $x^*(j) \notin \mathscr{E}(y(j))$ for some $j \in \operatorname{supp} y$ so $x^*(j) \notin \mathscr{E}(y(j))$ for some $j \in \operatorname{supp} x$. It follows that

$$\begin{aligned} x^*(x) &= \sum x^*(i)\, x(i) \leqslant \sum_{i \neq j} \| x(i)\| + \| x(j)\| - \epsilon_j \\ &= 1 - \epsilon_j \leqslant 1 - \epsilon, \end{aligned}$$

where $\epsilon = \min\{\epsilon_j : j \in \operatorname{supp} x \cap I_0\}$. Hence

$$\sup\{x^*(x): x^* \in \mathscr{E}(X^*)\backslash\mathscr{E}(y)\} \leqslant 1 - \epsilon < 1.$$

and the proof is complete.

Taking X_i to be the set of real numbers for each i, we deduce

COROLLARY 4.4. *$l_1(S)$ is an* (A)*-space for any set S.*

COROLLARY 4.5. *Let $L_1 = L_1(S, \Sigma, \mu)$, where (S, Σ, μ) is a σ-finite measure space. Then L_1 is an* (A)*-space if and only if (S, Σ, μ) is purely atomic.*

Proof. If L_1 is an (A)-space, then (by Theorem 2.4) L_1 is strongly nonlunar so, by Theorem 5.4 of [1], (S, Σ, μ) is purely atomic. Conversely, if (S, Σ, μ) is purely atomic, then L_1 is of type $l_1(T)$ for some set T, so the conclusion follows from Corollary 4.4.

REFERENCES

1. D. AMIR AND F. DEUTSCH, Suns, moons, and quasi-polyhedra, *J. Approximation Theory* **6** (1972), 176–201.
2. J. BLATTER, P. D. MORRIS, AND D. E. WULBERT, Continuity of the set-valued metric projection, *Math. Ann.* **178** (1968), 12–24.
3. B. BROSOWSKI, "Nichtlineare Tschebyscheff-Approximation," (BI-Hochschulskripten, Bd. 808/808a), Mannheim, 1968.
4. B. BROSOWSKI, Zur Charakterisierung von Minimallösungen im Raume $C[Q, H]$, *Bul. Inst. Politehnic Iaşi* **15** (XIX) (1969), 31–34.
5. B. BROSOWSKI, Nichtlineare Approximation in normierten Vektorräumen, "Abstract of Spaces and Approximation" (P. L. Butzer and B. Sz. Nagy, Eds.), (ISNM Vol. 10), Birkhäuser, Basel, 1969.
6. B. BROSOWSKI AND R. WEGMANN, Charakterisierung bester Approximationen in normierten Vektorräumen, *J. Approximation Theory* **3** (1970), 369–397.
7. B. BROSOWSKI AND R. WEGMANN, On the lower semi-continuity of the set-valued metric projection, *J. Approximation Theory* **8** (1973), 84–100.
8. M. DAY, "Normed Linear Spaces," Academic Press, New York, 1962.
9. N. DUNFORD AND J. T. SCHWARTZ, "Linear Operators," Part I, "General Theory," Interscience, New York, 1958.
10. N. EFIMOV AND S. STECKIN, Some properties of Chebyshev sets, *Dokl. Akad. Nauk SSSR* **118** (1958), 17–19 (Russian).
11. E. MICHAEL, Topologies on spaces of subsets, *Trans. Amer. Math. Soc.* **71** (1951), 152–182.
12. I. SINGER, Sur l'extension des fonctionnelles linéaires, *Rev. Math. Pures Appl.* **1** (1956), 99–106.
13. I. SINGER, Sur la meilleure approximation des fonctions abstraites continués à valeurs dans un espace de Banach, *Rev. Math. Pures Appl.* **2** (1957), 245–262.
14. L. VLASOV, On Chebyshev sets in Banach spaces, *Dokl. Akad. Nauk SSSR* **141** (1961), 19–20 (Russian).
15. H. WERNER, On the rational Tschebyscheff-operator, *Math. Z.* **86** (1964), 317–326.
16. W. WOLIBNER, Sur un polynome d'interpolation, *Colloq. Math.* **2** (1951), 136–137.